On a Class of Projectively Flat Finsler Metrics with Constant Flag Curvature

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One of important problems in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature. Another problem is to study and characterize projectively flat Finsler metrics on an open domain in R^n. In Riemannian geometry, these two problems are essentially same by Beltrami theorem. However, there are locally projectively flat Finsler metrics which are not of constant flag curvature; and there are Finsler metrics of constant flag curvature which are not locally projectively flat. It's a natural problem to discuss the projectively flat Finsler metrics with constant flag curvature. (α, β)-metrics is a computable class in Finsler metrics. It's interesting to classify the projectively flat (α, β)-metrics with constant flat curvature. In this lecture, I will introduce the recently result by Dr. Zhongmin Shen and me.

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The fundamental form $\mathbf{g}_y(u, v) = g_{ij}(x, y)u^i v^j$,

$$g_{ij} = \left[\frac{1}{2}F^2\right]_{y^i y^j} > 0.$$

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Geodesics:
$\ddot{x} + 2G^i(x, \dot{x}) = 0$, where $G^i = G^i(x, y)$ are given by

$$G^i = \frac{1}{4}g^{il}\left\{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\right\}.$$

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Riemann curvature: $\mathbf{R}_y(u) = R^i{}_k(x,y)u^k$,

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

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Flag curvature:

$$\mathbf{K} = \mathbf{K}(P,y) = \frac{\mathbf{g}_y(\mathbf{R}_y(u),u)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,y) - [\mathbf{g}_y(y,u)]^2},$$

where $P = \mathrm{span}\{y,u\} \subset T_xM$.

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where $P = \mathrm{span}\{y,u\} \subset T_xM$.
For a Riemannian metric $F = \sqrt{g_{ij}(x)y^iy^j}$, $\mathbf{g}_y = \mathbf{g}$,

$$\mathbf{R}_y(u) = R(u,y)y.$$

$$\mathbf{g}(\mathbf{R}_y(u),u) = \mathbf{g}(R(u,y)y,u) = \mathbf{g}(R(y,u)u,y) = \mathbf{g}(\mathbf{R}_u(y),y).$$

$$\mathbf{K}(P,y) = \mathbf{K}(P) \quad \text{(sectional curvature)}.$$

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Another equivalent condition of projectively flat Finsler metric $F = F(x,y)$ on an open subset $\mathcal{U} \subset R^n$,

$$F_{x^k y^l} y^k - F_{x^l} = 0.$$

This is by G. Hamel.

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The flag curvature of projectively flat Finsler metrics:

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The flag curvature of projectively flat Finsler metrics:
If $G^i = P(x,y)y^i$, then

$$\mathbf{K} = \frac{P^2 - P_{x^m} y^m}{F^2}.$$

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where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. $\phi = \phi(s)$ is a C^∞ function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(0) = 1, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,$$

where $b = \|\beta_x\|_\alpha$.

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$\phi(s) = 1/(1 - s)$, $F = \alpha^2/(\alpha - \beta)$, Matsumoto metric.

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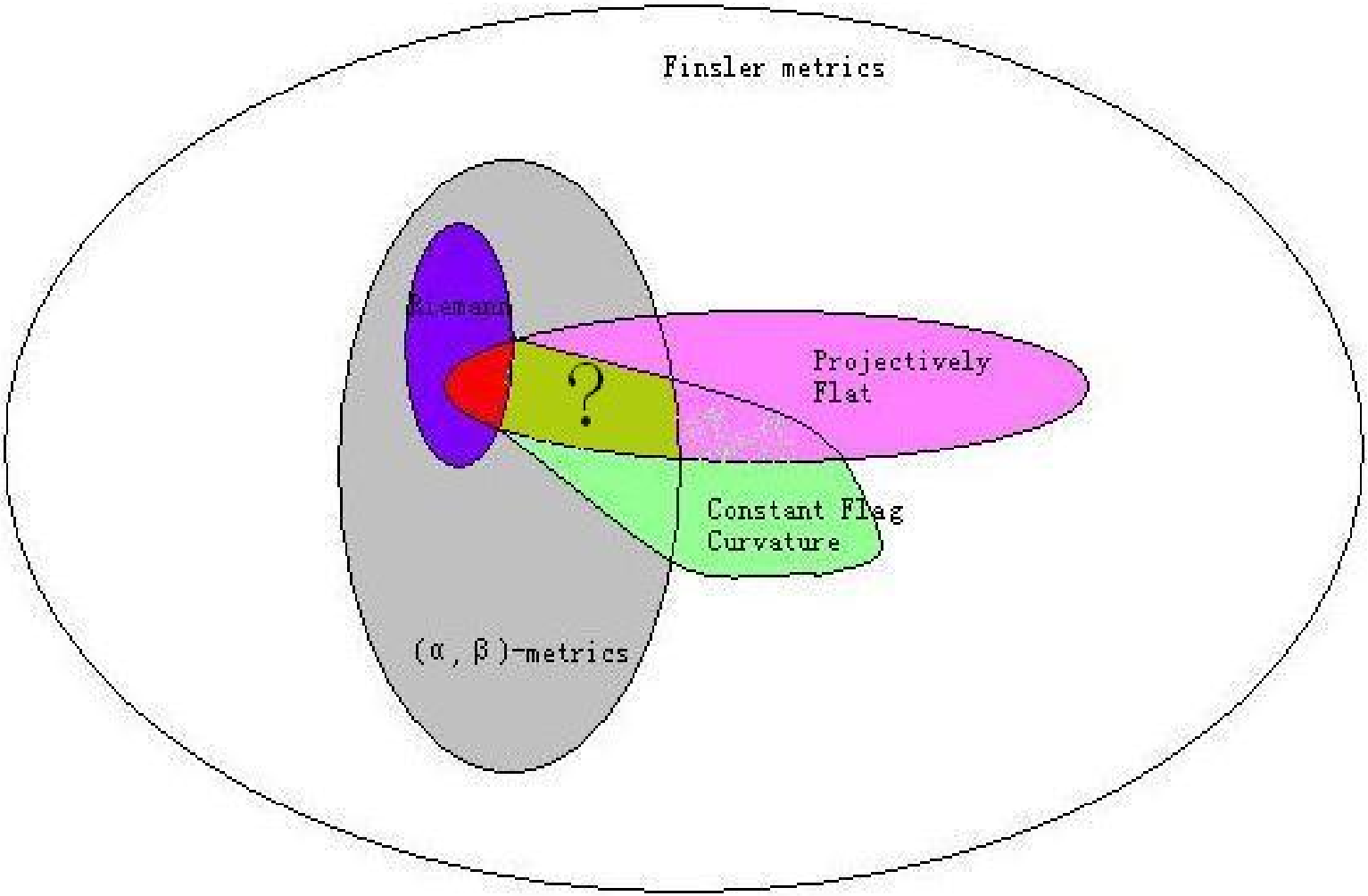

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$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \alpha^{-1}\Theta\Big(-2\alpha Q s_0 + r_{00}\Big)y^i + \Psi\Big(-2\alpha Q s_0 + r_{00}\Big)b^i,$$

where G^i_α is the geodesic coefficient of α and $s_{ij} = b_{i|j} - b_{j|i}$, $r_{ij} = b_{i|j} + b_{j|i}$, $s_{i0} = s_{ij}y^j$, $s_0 = s_{i0}b^i$, $r_{00} = r_{ij}y^iy^j$ and

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'} \\ \Theta &= \frac{\phi - s\phi'}{2\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \cdot \frac{\phi'}{\phi} - s\Psi \\ \Psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

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Proposition *A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed.*

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Proposition *A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed.*

Theorem(Shen-Yildirim) *Let $(\alpha + \beta)^2/\alpha$ be a Finsler metric on a manifold M. F is projectively flat if and only if*

(i) $b_{i|j} = \tau\{(1 + 2b^2)a_{ij} - 3b_i b_j\}$,

(ii) *the geodesic coefficients G^i_α of α are in the form:* $G^i_\alpha = \theta y^i - \tau\alpha^2 b^i$,

where $\tau = \tau(x)$ is a scalar function and $\theta = a_i(x)y^i$ is a 1-form on M.

Theorem (Shen) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space R^n $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i \neq 0$. Suppose that the following conditions: (a) β is not parallel with respect to α, (b) F is not of Randers type, and (c) $db \neq 0$ everywhere or $b =$ constant on $\mathcal{U}$. Then F is projectively flat on $\mathcal{U}$ if and only if the function $\phi = \phi(s)$ satisfies*

$$\Big\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big\}\phi''(s) = (k_1 + k_2 s^2)\Big\{\phi(s) - s\phi'(s)\Big\},$$

$$b_{i|j} = 2\tau\Big\{(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\Big\},$$

$$G^i_\alpha = \xi y^i - \tau\Big(k_1\alpha^2 + k_2\beta^2\Big)b^i,$$

where $\tau = \tau(x)$ is a scalar function on $\mathcal{U}$ and k_1, k_2 and k_3 are constants.

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Examples: Projectively Flat Finsler Metrics with Constant Flag Curvature

$\mathbf{K} = -\frac{\mathbf{1}}{\mathbf{4}}$. Funk metric Θ on the unit ball $B^n \subset R^n$:

$$\Theta = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2} + \frac{\langle x, y\rangle}{1-|x|^2},$$

where $y \in T_x \mathrm{B}^n \approx R^n$.

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where $y \in T_x \mathrm{B}^n \approx R^n$.
$\mathbf{K} = \mathbf{0}$. Berwald's metric

$$B = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}},$$

where $y \in T_x \mathrm{B}^n \approx R^n$.

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The Funk metric and Berwald's metric are related and they can be expressed in the form

$$\Theta = \bar{\alpha} + \bar{\beta}, \qquad B = \frac{(\tilde{\alpha} + \tilde{\beta})^2}{\tilde{\alpha}},$$

where

$$\bar{\alpha} := \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2}, \qquad \bar{\beta} := \frac{\langle x, y\rangle}{1-|x|^2},$$

$$\tilde{\alpha} := \lambda\bar{\alpha}, \qquad \tilde{\beta} := \lambda\bar{\beta}, \qquad \lambda := \frac{1}{1-|x|^2}.$$

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Lemma *If*
$\phi(s) = 1+a_1 s+a_2 s^2+a_3 s^3+a_4 s^4+a_5 s^5+a_6 s^6+a_7 s^7+a_8 s^8+o(s^8)$
satisfies
$\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\}\phi''(s) = (k_1 + k_2 s^2)\{\phi(s) - s\phi'(s)\},$
then

$$a_2 = \frac{k_1}{2}, \quad a_3 = 0, \quad a_5 = 0, \quad a_7 = 0,$$

$$\begin{aligned}
a_4 &= \frac{1}{12}(k_2 - k_1 k_3) - \frac{1}{8}k_1^2, \\
a_6 &= -\frac{11}{120}(k_1 + \frac{4}{11}k_3)(k_2 - k_1 k_3) + \frac{1}{16}k_1^3, \\
a_8 &= \frac{1}{56}(k_2 - k_1 k_3)(\frac{61}{12}k_1^2 + k_3^2) - \frac{5}{224}k_2^2 \\
&\quad + \frac{31}{336}k_1 k_2 k_3 - \frac{47}{672}k_1^2 k_3^2 - \frac{5}{128}k_1^4.
\end{aligned}$$

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Lemma (Li-Shen) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U} \subset R^n$ $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i \neq 0$. Suppose that F is not of Randers type and $db \neq 0$ everywhere or $b = constant$ on $\mathcal{U}$. If F is projectively flat with constant flag curvature K, then $K = 0$.*

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Sketch proof:

- β is parallel with respect to α.

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Then $G^i = G^i_\alpha$, which means α is projectively flat. By Beltrami theorem α has constant sectional curvature κ. If $\kappa \neq 0$, it's easy to see that F is a Riemannian metric. This is excluded. Thus $K = 0$.

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- β is not parallel with respect to α.

By a direct computation we get

$$K\alpha^2\phi^2 = \xi^2 - \xi_{x_m}y^m + \tau^2\alpha^2\Xi - \tau_{x_m}y^m\Xi^2 + 2\tau^2\alpha^2\Gamma,$$

where $\xi = \xi_i y^i$, $\tau = \tau(x)$,

$$\Xi := (1 + (k_1 + k_2 s^2)s^2 + k_3 s^2)\frac{\phi'}{\phi} - (k_1 + k_2 s^2)s,$$

$$\Gamma := (k_1 + k_2 s^2)s\Xi - \left\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\right\}\Xi_s.$$

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Special coordinate system: This is first used by Z. Shen.

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Special coordinate system: This is first used by Z. Shen. Fix an arbitrary point $x_o \in \mathcal{U} \subset R^n$. Make a change of coordinates: $(s, y^a) \to (y^i)$ by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^a = y^a,$$

where $\bar{\alpha} := \sqrt{\sum_{a=2}^{n}(y^a)^2}$. Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}.$$

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where $\bar{\alpha} := \sqrt{\sum_{a=2}^{n}(y^a)^2}$. Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}.$$

And

$$\xi = \frac{s\xi_1}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{\xi}_0, \quad \tau_{x^m}y^m = \frac{s\tau_1}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{\tau}_0,$$

where $\bar{\xi}_0 := \xi_a y^a$, $\bar{\tau}_0 := \tau_{x^a}y^a$. Let

$$\xi_{ij} := \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x^j} + \frac{\partial \xi_j}{\partial x^i}\right).$$

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By above identities, we obtain that

$$\begin{aligned}\frac{Kb^2\phi^2}{b^2-s^2}\bar{\alpha}^2 &= \frac{\bar{\alpha}^2}{b^2-s^2}\Big\{s^2(\xi_1^2-\xi_{11})+\tau^2b^2\Xi^2-\tau_1bs\Xi+2\tau^2b^2\Gamma\Big\}\\ &\quad+\frac{1}{\sqrt{b^2-s^2}}\Big\{2s\xi_1\bar{\xi}_0-2s\bar{\xi}_{10}-\bar{\tau}_0b\Xi\Big\}\bar{\alpha}+\bar{\xi}_0^2-\bar{\xi}_{00}.\end{aligned}$$

The above equation is equivalent to the two following equations

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The above equation is equivalent to the two following equations

$$2s(\xi_1\bar{\xi}_0-\bar{\xi}_{10})-\bar{\tau}_0b\Xi=0,$$

$$\begin{aligned}&\frac{1}{b^2-s^2}\Big\{s^2(\xi_1^2-\xi_{11})+\tau^2b^2\Xi^2-\tau_1bs\Xi+2\tau^2b^2\Gamma-Kb^2\phi^2\Big\}\bar{\alpha}^2\\ &+\bar{\xi}_0^2-\bar{\xi}_{00}=0.\end{aligned}$$

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By Taylor expansion of ϕ

$$\phi=1+a_1s+a_2s^2+a_4s^4+a_6s^6+a_8s^8+o(s^8)$$

we eventually proof $K=0$.

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we eventually proof $K=0$.

Remark: In this Lemma, we used the equivalent conditions of projectively flat (α,β)-metrics.

Projectively Flat (α, β)-Metrics with Zero Flag Curvature

Lemma(Li-Shen) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U} \subset R^n$ $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i \neq 0$. Suppose that F is not of Randers type, β is not parallel with respect to α and $db \neq 0$ everywhere or $b =$ constant on $\mathcal{U}$. If F is projectively flat with $K = 0$, then*

$$\phi = \frac{(\sqrt{1 + ks^2} + \epsilon s)^2}{\sqrt{1 + ks^2}},$$

where $k = \frac{1}{5}(3k_1 + 2k_3)$ and $\epsilon = \pm\frac{2}{\sqrt{5}}\sqrt{k_1 - k_3}$.

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Sketch proof: By assumption that $K = 0$, we have

$$3\Big\{ - a_1^4 s^2 - (1 + 2k_3 s^2 + k_1 s^2 + 2k_2 s^4)a_1^2 + (k_1 + k_2 s^2)^2 s^2 \Big\}\tau^2\phi^2$$
$$+ 6\Big\{ s a_1^2 - s(k_1 + k_2 s^2) \Big\} D(s)\tau^2\phi'\phi + 3\tau^2 D(s)^2\phi'^2 = 0,$$

where $D(s) = 1 + (k_1 + k_2 s^2)s^2 + k_3 s^2$. By discuss this equation we get the result.

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By the above two lemmas we obtain the following.

Theorem(Li-Shen) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space R^n $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i \neq 0$. Suppose that $db \neq 0$ everywhere or $b =$ constant on $\mathcal{U}$. Then F is projectively flat with constant flag curvature K if and only if one of the following holds*

(i) *α is projectively flat and β is parallel with respect to α;*

(ii) *$F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta$ is projectively flat with constant flag curvature $K < 0$, where k and $\epsilon \neq 0$ are constants;*

(iii) *$F = (\sqrt{\alpha^2 + k\beta^2} + \epsilon\beta)^2/\sqrt{\alpha^2 + k\beta^2}$ is projectively flat with $K = 0$, where k and $\epsilon \neq 0$ are constants.*

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(iii) *$F = (\sqrt{\alpha^2 + k\beta^2} + \epsilon\beta)^2/\sqrt{\alpha^2 + k\beta^2}$ is projectively flat with $K = 0$, where k and $\epsilon \neq 0$ are constants.*

• It is a trivial fact that if F is trivial and the flag curvature $K = constant$, then it is either Riemannian $(K \neq 0)$ or locally Minkowskian $(K = 0)$. (by S. Numata).

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- The Finsler metric in (ii) is of Randers type, i.e.,

$$F = \bar{\alpha} + \bar{\beta},$$

where $\bar{\alpha} := \sqrt{\alpha^2 + k\beta^2}$ and $\bar{\beta} := \epsilon\beta$.
Shen proved that a Finsler metric in this form is projectively flat with constant flag curvature if and only if it is locally Minkowskian or it is locally isometric to a generalized Funk metric

$$F = c(\bar{\alpha} + \bar{\beta})$$

on the unit ball $B^n \subset R^n$, where $c > 0$ is a constant, and

$$\bar{\alpha} : = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2} \tag{1}$$

$$\bar{\beta} : = \pm\Big\{\frac{\langle x, y\rangle}{1-|x|^2} + \frac{\langle a, y\rangle}{1+\langle a, x\rangle}\Big\}, \tag{2}$$

where $a \in R^n$ is a constant vector.

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- The Finsler metric in (iii) is in the form

$$F = (\tilde{\alpha} + \tilde{\beta})^2/\tilde{\alpha},$$

where $\tilde{\alpha} := \sqrt{\alpha^2 + k\beta^2}$ and $\tilde{\beta} := \epsilon\beta$. It is proved (by Mo, Shen , Yang and Yildirim) that a non-Minkowkian metric $F = (\tilde{\alpha} + \tilde{\beta})^2/\tilde{\alpha}$ is projectively flat with $K = 0$ if and only if it is, after scaling on x, locally isometric to a metric

$$F = c(\tilde{\alpha} + \tilde{\beta})^2/\tilde{\alpha}$$

on the unit ball $B^n \subset R^n$, where $c = constant$, $\tilde{\alpha} = \lambda\bar{\alpha}$ and $\tilde{\beta} = \lambda\bar{\beta}$, where $\bar{\alpha}$ and $\bar{\beta}$ are given in (1) and (2), and

$$\lambda := \frac{(1 + \langle a, x\rangle)^2}{1 - |x|^2}.$$

An Open Problem

Is there any metric $F = (\alpha + \beta)^2/\alpha$ of constant flag curvature which is not locally projectively flat?

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Indicatrix of $F = (\alpha + \beta)^2/\alpha$

Indicatrix of F: Given a Minkowski space (V, F),

$$S_F := \Big\{ y \in V | F(y) = 1 \Big\}.$$

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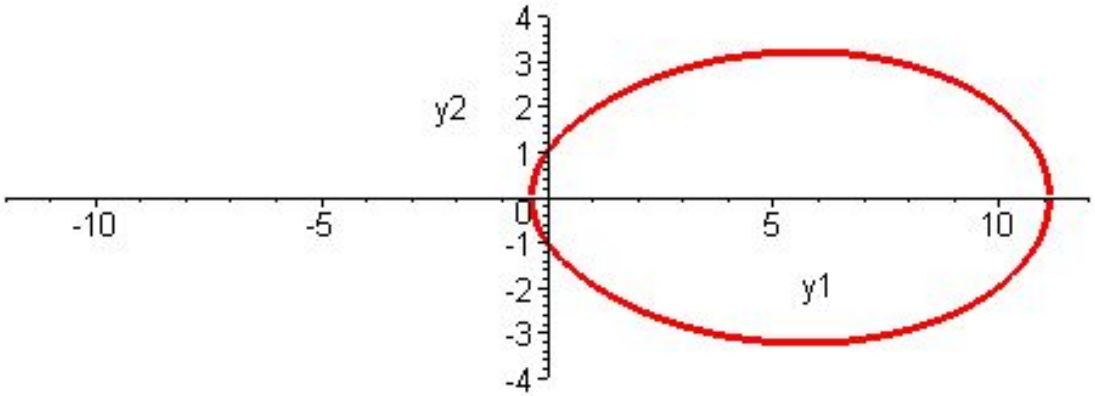
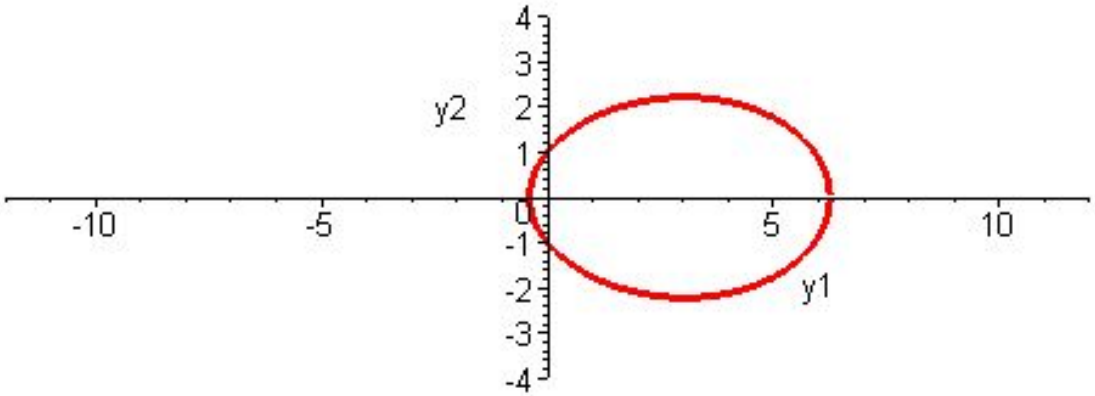
$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



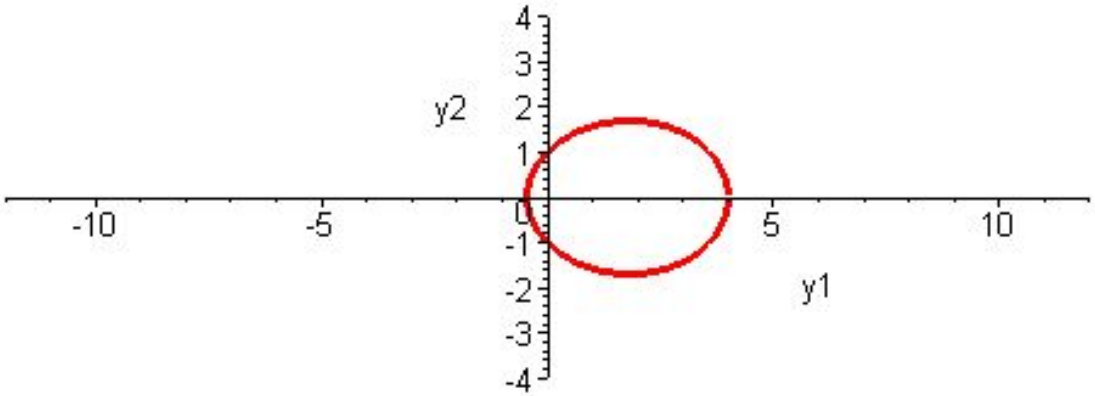
$b = -0.7$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



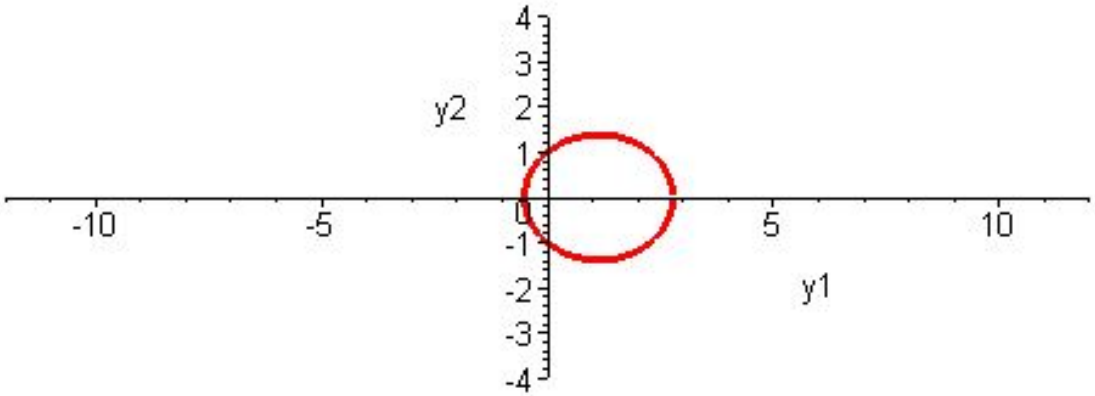
$b = -0.6$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



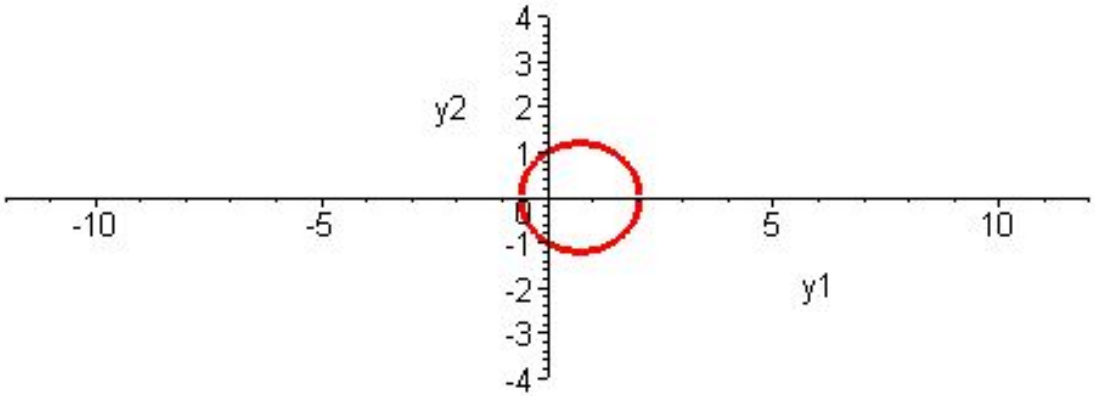
$b = -0.5$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



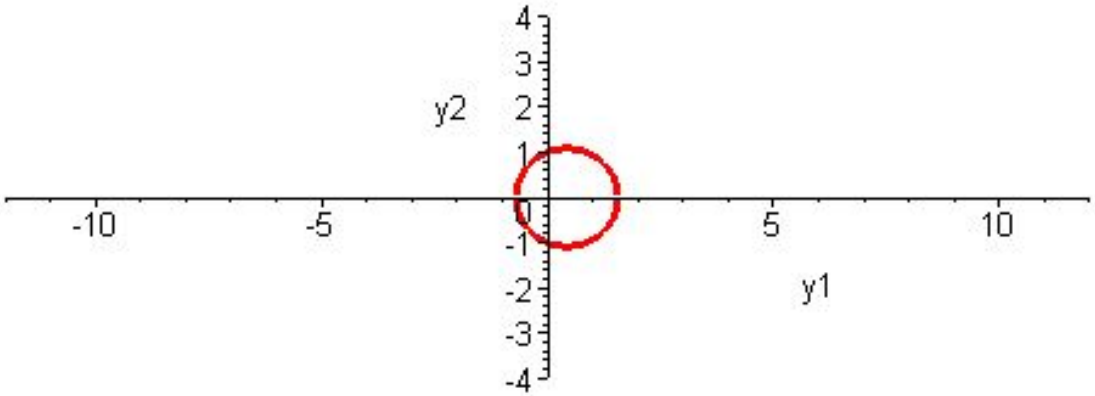
$b = -0.4$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



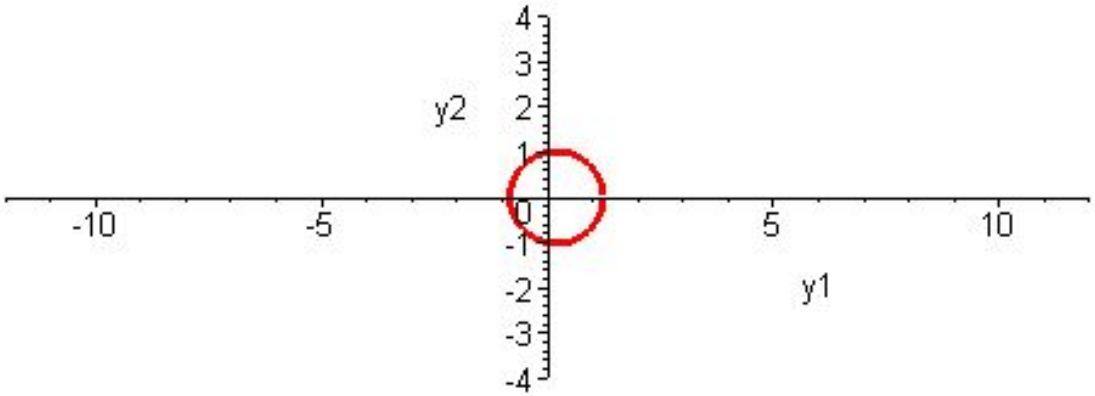
$b = -0.3$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



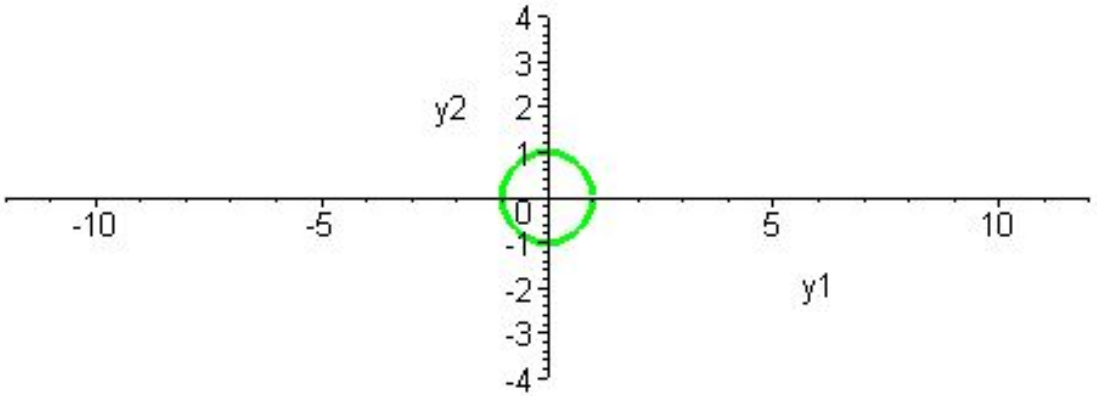
$b = -0.2$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



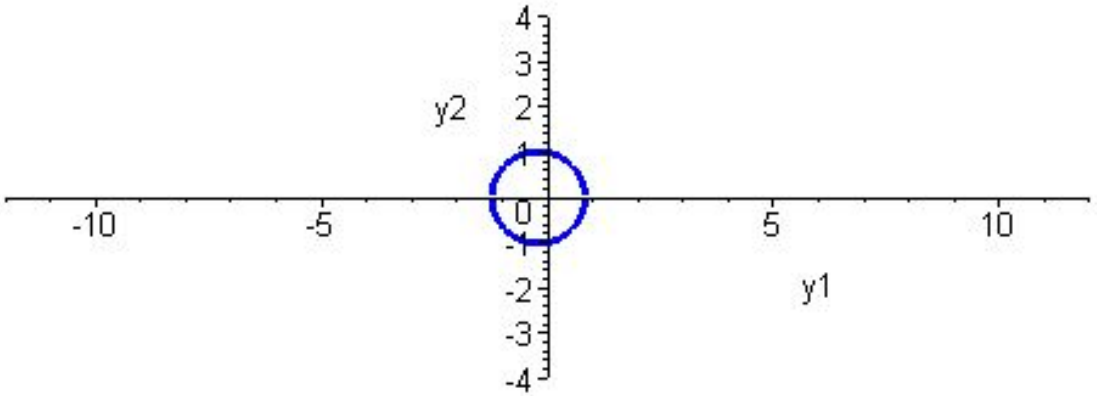
$b = -0.1$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



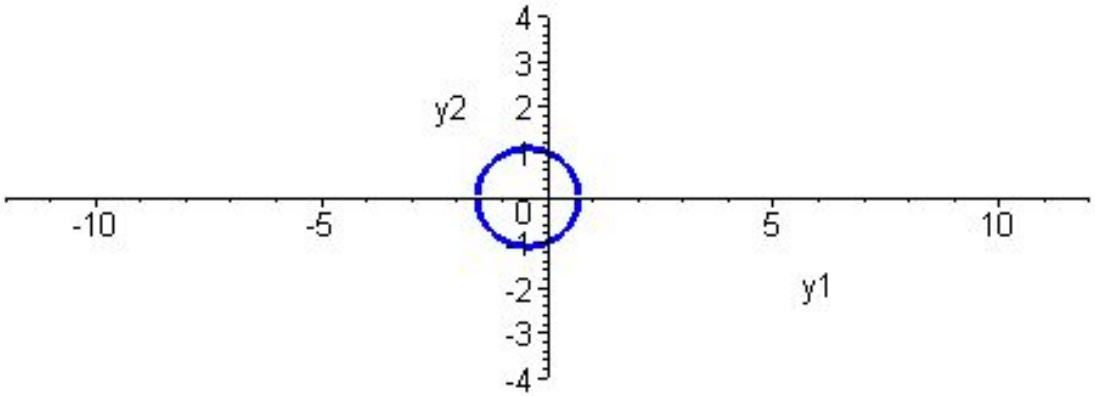
$b = 0$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



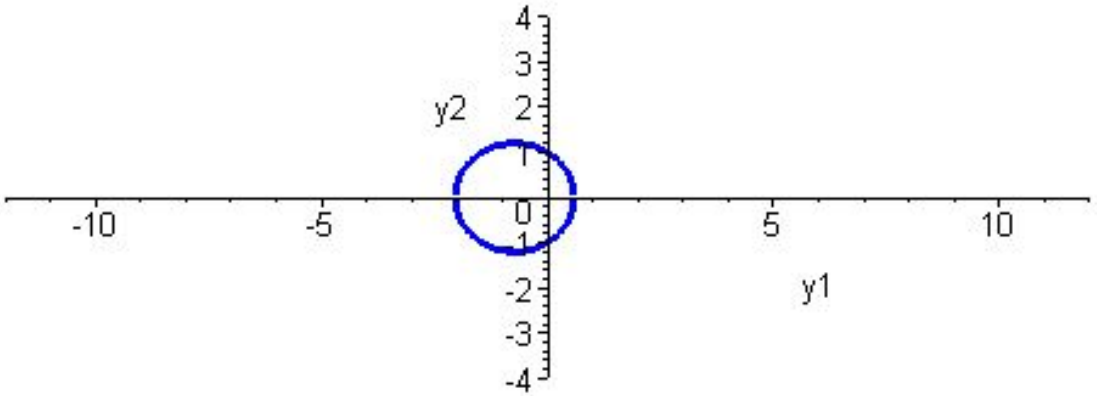
$b = 0.1$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



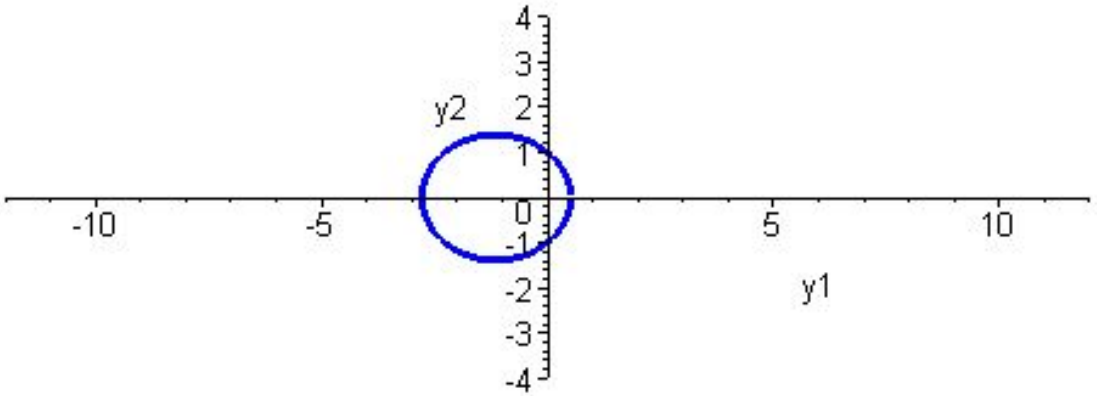
$b = 0.2$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



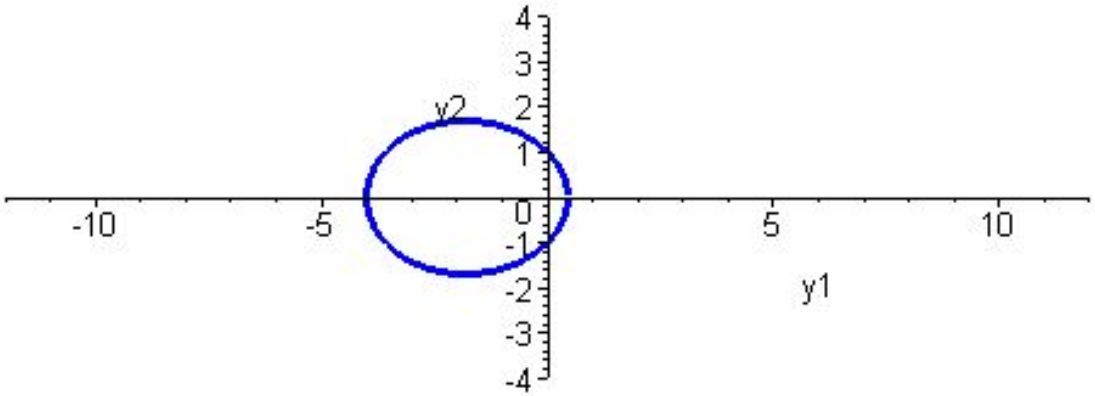
$b = 0.3$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



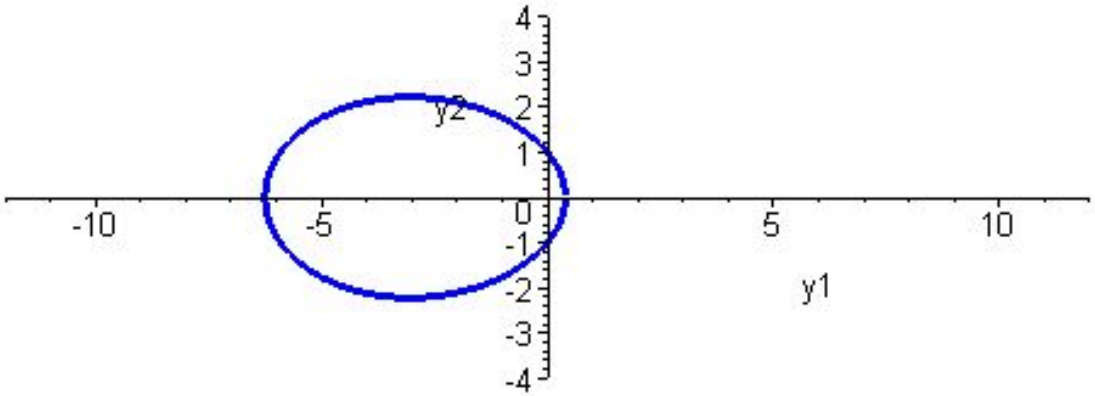
$b = 0.4$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



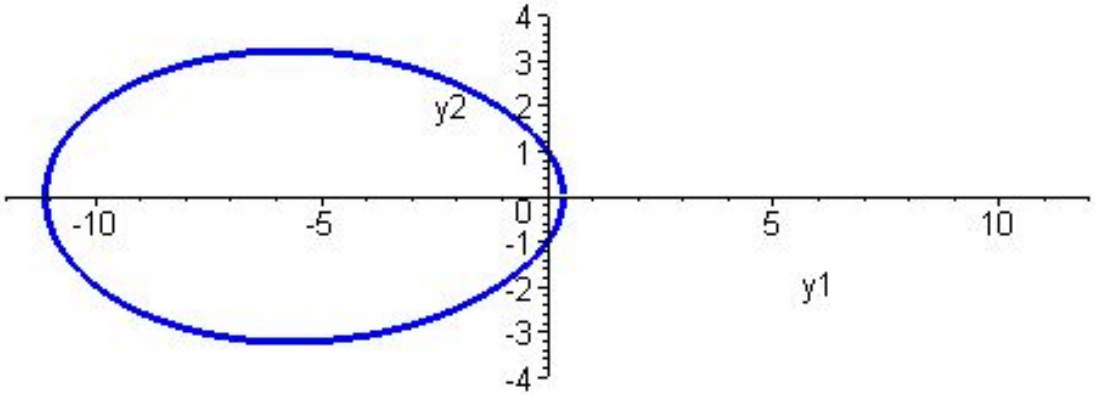
$b = 0.5$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



$b = 0.6$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



$b = 0.7$

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Thank you very much!

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